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On An Alternative Supermatrix Reduction

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Abstract. We consider a nonstandard odd reduction of supermatrices (as compared with the standard even reduction) which arises in connection with the possible extension of manifold structure group reductions. The study was initiated by consideration of generalized noninvertible superconformal-like transformations. The features of even- and odd-reduced supermatrices are investigated together. They can be unified into some kind of 'sandwich' semigroups. We also define a special module over even- and odd-reduced supermatrix sets, and the generalized Cayley–Hamilton theorem is proved for them. It is shown that the odd-reduced supermatrices represent semigroup bands and Rees matrix semigroups over a unit group.

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1. Introduction

According to the general theory of G-structures [1], various geometries are obtained by the reduction of a structure group of a manifold to some subgroup G of tangent space endomorphisms. In the local approach using a coordinate description, this means that one should reduce a corresponding matrix in a given representation to a reduced form as a matter of fact. In most cases, this form is triangular, and one of the reasons for this is the transparent observation from ordinary matrix theory that triangular matrices preserve the shape and form of a subgroup. In supersymmetric theories, despite the appearance of odd subspaces and anticommuting variables, the choice of the reduction shape remains the same [2], and one ground for this is the full identity of the supermatrix multiplication with the ordinary one and, consequently, it can be naively assumed that the shape of the matrices that form a substructure should be the same. In an exacting search of nontrivial supersymmetric manifestations, one can observe that the closure of multiplication can be also achieved for other shapes, due to the existence of zero divisors in the Grassmann algebra or in the ring over which a theory is defined. So the meaning of the reduction itself can be extended in principle.

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This Letter was initiated by the study of superconformal symmetry semigroup extensions [3–5]. Indeed, superconformal transformations [6] appear as a result of the reduction of a structure group matrix to a triangular form [7]. Also, the transition functions on semirigid surfaces [8] which are used to describe topological supergravity, have the same origin. In [3], we considered an alternative version of the reduction. The superconformal-like transformations obtained in this way have many unusual features, e.g. they are noninvertible and twist parity of the tangent space in the supersymmetric basis. We note that this situation substantially differs from the case of Q-manifolds [9], where changing the parity of the tangent space is done by hand from the first definitions.

Here, we study an alternative reduction of supermatrices from a more abstract viewpoint without connecting it to a special physical model.

2. Preliminaries

A $(p|q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ (in the sense of [10]) is the even sector of the direct product $\Lambda^{p|q} = \Lambda_0^p \times \Lambda_1^q$, where Λ is a commutative Banach $\mathbb{Z}_2$-graded superalgebra [11,12] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| =_{\text{def}} \{i \in \{0,1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$. The even homomorphism $\mathfrak{m}_b: \Lambda \to \mathbb{B}$, where $\mathbb{B}$ is a purely even algebra over $\mathbb{K}$, is called a body map [10]. If there exists an embedding $\mathfrak{n}: \mathbb{B} \hookrightarrow \Lambda$ such that $\mathfrak{m}_b \circ \mathfrak{n} = \text{id}$, then Λ admits the body and soul decomposition $\Lambda = \mathbb{B} \oplus \mathbb{S}$, and a soul map can be defined as $\mathfrak{m}_s: \Lambda \to \mathbb{S}$. Usually, the isomorphism $\mathbb{B} \cong \mathbb{K}$ is implied (which is not generally necessary and can lead to nontrivial behavior of the body). In case where Λ is a Banach algebra (with a norm $||\cdot||$) soul elements are quasinilpotent [13], which means $\forall a \in \mathbb{S}, \lim_{n\to\infty} ||a||^{1/n} = 0$. But quasinilpotency of the soul elements does not necessarily lead to their nilpotency ($\forall a \in \mathbb{S}\, \exists n,\, a^n = 0$) for the infinite-dimensional case [14]. These facts allow us to consider noninvertible morphisms on a par with invertible ones (in some sense), which gives, in proper conditions, many interesting and nontrivial results (see [3,15]).

The even morphisms $\text{Hom}_0(\Lambda^{p|q}, \Lambda^{m|n})$ between superlinear spaces $\Lambda^{p|q} \to \Lambda^{m|n}$ are described by means of $(m+n) \times (p+q)$-supermatrices (see [11]). In the theory of super-Riemann surfaces [6], the $(1+1) \times (1+1)$-supermatrices describing holomorphic morphisms of the tangent bundle have a triangle shape [7]. Here we consider a special alternative reduction of supermatrices. For transparency and clarity, we confine ourselves to $(1+1) \times (1+1)$-supermatrices, which will allow us to clarify ideas without hiding them behind large formulas. We also omit some evident proofs. The generalization to the $(m+n) \times (p+q)$ case is straightforward and can be mostly done by means of simple notation changing.

3. Structure of $\mathrm{Mat}_\Lambda(1|1)$

In the standard basis, the elements from $\mathrm{Hom}_0(\Lambda^{1|1}, \Lambda^{1|1})$ are described by the $(1+1) \times (1+1)$-supermatrices [11]

$$M \equiv \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{Mat}_\Lambda(1|1),$$

where $a, b \in \Lambda_0$, $\alpha, \beta \in \Lambda_1$ (in the following, we use Latin letters for elements from Λ_0 and Greek letters for ones from Λ_1, and we suppose that odd elements are of second nilpotency degree). For sets of supermatrices, we also use corresponding bold symbols, e.g. $\mathbf{M} =_{\mathrm{def}} \{M \in \mathrm{Mat}_\Lambda(1|1)\}$. In this simple $(1|1)$ case, the Berezinian [11] defined as $\mathrm{Ber} : \mathrm{Mat}_\Lambda(1|1) \setminus \{M | \, \mathrm{m}_b(b) = 0\} \to \Lambda_0$ is

$$\mathrm{Ber}\, M = \frac{a}{b} + \frac{\beta\alpha}{b^2}. \tag{1}$$

Now we define two types of possible reductions of M together and study some of their properties simultaneously.

DEFINITION 1. Even-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda(1|1)$ having the form

$$S \equiv \begin{pmatrix} a & \alpha \\ 0 & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{\mathrm{even}}(1|1). \tag{2}$$

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda(1|1)$ having the form

$$T \equiv \begin{pmatrix} 0 & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{\mathrm{odd}}(1|1). \tag{3}$$

The explanation for the basis of the notations comes from the nilpotency of $\mathrm{Ber}\, T$ and from the fact that the even-reduced supermatrices S give superconformal transformations which describe morphisms of the tangent bundle over the super-Riemann surfaces [7], while the odd-reduced supermatrices T give the superconformal-like transformations twisting the parity of the $(1|1)$ tangent superspace in the standard basis (see [3, 5]).

ASSERTION 2. $\mathbf{M}$ *is a direct sum of diagonal* $\mathbf{D}$ *and anti-diagonal (secondary diagonal)* $\mathbf{A}$ *supermatrices (the even and odd ones in the notations of* [11]*)*

$$\mathbf{M} = \mathbf{D} \oplus \mathbf{A}, \tag{4}$$

where

$$D \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbf{D} \equiv \mathrm{Mat}_\Lambda^{\mathrm{Diag}}(1|1),$$

$$A \equiv \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \mathbf{A} \equiv \mathrm{Mat}_{\Lambda}^{\mathrm{Adiag}}(1|1),$$

and $\mathbf{D} \subset \mathbf{S}$ *and* $\mathbf{A} \subset \mathbf{T}$.

For the reduced supermatrices, one finds

$$\mathbf{S} \cap \mathbf{T} = \begin{pmatrix} 0 & \alpha \\ 0 & b \end{pmatrix} \neq \emptyset. \tag{5}$$

Nevertheless, the following observation explains the fundamental and dual roles of the even-reduced supermatrices $\mathbf{S}$ and the odd-reduced ones $\mathbf{T}$.

THEOREM 3 (Berezinian addition theorem). *The Berezinians of even- and odd-reduced supermatrices are additive components of the Berezinian of the corresponding nonreduced supermatrix*

$$\mathrm{Ber}\, M = \mathrm{Ber}\, S + \mathrm{Ber}\, T. \tag{6}$$

The first term in (6) covers all subgroups of the even-reduced supermatrices from $\mathrm{Mat}_{\Lambda}(1|1)$, and only later was it considered in applications. But the second term is dual to the first in some sense and corresponds to all subsemigroups of odd-reduced supermatrices from $\mathrm{Mat}_{\Lambda}(1|1)$ (the relation (6) is a supersymmetric version of the obvious equality $\det M = \det D + \det A$, where D and A from (4) are here ordinary matrices, but the problem is that for A being a supermatrix, $\mathrm{Ber}\, A$ is not defined at all).

Denote a set of invertible elements of $\mathbf{M}$ by $\mathbf{M}^*$, and $\mathbf{I} = \mathbf{M} \setminus \mathbf{M}^*$. In [11], it was proved that $\mathbf{M}^* = \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) \neq 0 \wedge \mathfrak{m}_b(b) \neq 0\}$. Then, similarly, $\mathbf{S}^* = \{S \in \mathbf{S} | \, \mathfrak{m}_b(a) \neq 0 \wedge \mathfrak{m}_b(b) \neq 0\}$ and $\mathbf{T}^* = \emptyset$, i.e. the odd-reduced matrices are noninvertible and $\mathbf{T} \subset \mathbf{I}$. Consider the invertibility structure of $\mathrm{Mat}_{\Lambda}(1|1)$ in more detail. Let us denote

$$\begin{aligned} \mathbf{M}' &= \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) \neq 0\}, \qquad & \mathbf{M}'' &= \{M \in \mathbf{M} | \, \mathfrak{m}_b(b) \neq 0\}, \\ \mathbf{I}' &= \{M \in \mathbf{M} | \, \mathfrak{m}_b(a) = 0\}, \qquad & \mathbf{I}'' &= \{M \in \mathbf{M} | \, \mathfrak{m}_b(b) = 0\}. \end{aligned} \tag{7}$$

Then

$$\mathbf{M} = \mathbf{M}' \cup \mathbf{I}' = \mathbf{M}'' \cup \mathbf{I}'' \quad \text{and} \quad \mathbf{M}' \cap \mathbf{I}' = \emptyset, \mathbf{M}'' \cap \mathbf{I}'' = \emptyset,$$

therefore $\mathbf{M}^* = \mathbf{M}' \cap \mathbf{M}''$ and $\mathbf{T} \subset \mathbf{M}''$. The Berezinian $\mathrm{Ber}\, M$ is well-defined for the supermatrices from $\mathbf{M}''$ only and is invertible when $M \in \mathbf{M}^*$, but for the supermatrices from $\mathbf{M}'$, the inverse $(\mathrm{Ber}\, M)^{-1}$ is well-defined and is invertible when $M \in \mathbf{M}^*$ too [11].

Under the ordinary supermatrix multiplication, the set $\mathbf{M}$ is a semigroup of all $(1|1)$ supermatrices [16], and the set $\mathbf{M}^*$ is a subgroup of $\mathbf{M}$. In the standard basis, $\mathbf{M}^*$ represents the general linear group $\mathrm{GL}_{\Lambda}(1|1)$ [11]. A subset $\mathbf{I} \subset \mathbf{M}$ is an ideal of the semigroup $\mathbf{M}$ [17].

PROPOSITION 4. (1) *The sets* $\mathbf{I}$, $\mathbf{I}'$ *and* $\mathbf{I}'$ *are isolated ideals of* $\mathbf{M}$.

(2) *The sets* $\mathbf{M}^*$, $\mathbf{M}'$ *and* $\mathbf{M}''$ *are filters of the semigroup* $\mathbf{M}$.

(3) *The sets* $\mathbf{M}'$ *and* $\mathbf{M}''$ *are subsemigroups of* $\mathbf{M}$, *which are* $\mathbf{M}' = \mathbf{M}^* \cup \mathbf{J}'$ *and* $\mathbf{M}'' = \mathbf{M}^* \cup \mathbf{J}''$ *with the isolated ideals*

$$\mathbf{J}' = \mathbf{M}' \setminus \mathbf{M}^* = \mathbf{M}' \cap \mathbf{I}'' \quad \text{and} \quad \mathbf{J}'' = \mathbf{M}'' \setminus \mathbf{M}^* = \mathbf{M}'' \cap \mathbf{I}'$$

respectively (cf. [11], pp. 95, 103).

(4) *The ideal of the semigroup* $\mathbf{M}$ *is* $\mathbf{I} = \mathbf{I}' \cup \mathbf{J}' = \mathbf{I}'' \cup \mathbf{J}''$.

Proof. Let $M_3 = M_1 M_2$, then $a_3 = a_1 a_2 + \alpha_1 \beta_2$ and $b_3 = b_1 b_2 + \beta_1 \alpha_2$. Taking the body part, we derive $\mathfrak{m}_b(a_3) = \mathfrak{m}_b(a_1)\,\mathfrak{m}_b(a_2)$, and $\mathfrak{m}_b(b_3) = \mathfrak{m}_b(b_1)\,\mathfrak{m}_b(b_2)$. Then use the definitions. □

4. Multiplication Properties of Odd-Reduced Supermatrices

The odd-reduced supermatrices do not form a semigroup in the general case, since

$$T_1 T_2 = \begin{pmatrix} \alpha_1 \beta_2 & \alpha_1 b_2 \\ b_1 \beta_2 & b_1 b_2 + \beta_1 \alpha_2 \end{pmatrix} \neq T.$$

However, it follows

$$\mathbf{T} \cdot \mathbf{T} \cap \mathbf{T} \neq \emptyset \Rightarrow \alpha\beta = 0, \qquad \mathbf{T} \cdot \mathbf{T} \cap \mathbf{S} \neq \emptyset \Rightarrow \beta b = 0, \tag{8}$$

that can take place, because of the existence of zero divisors in Λ. In (8), the point denotes the standard supermatrix set multiplication:

$$\mathbf{A} \cdot \mathbf{B} \underset{\text{def}}{=} \{\cup AB \mid A \in \mathbf{A}, B \in \mathbf{B}\}.$$

PROPOSITION 5. (1) *The subset* $\mathbf{T}^{\mathrm{SG}} \subset \mathbf{T}$ *of the odd-reduced supermatrices satisfying* $\alpha\beta = 0$ *form an odd-reduced subsemigroup of* $\mathbf{M}$.

(2) *In the odd-reduced semigroup* $\mathbf{T}^{\mathrm{SG}}$, *the subset of supermatrices with* $\beta = 0$ *is a left ideal, and one with* $\alpha = 0$ *is a right ideal, the matrices with* $b = 0$ *form a two-sided ideal.*

Let

$$Z_\alpha(t) = \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \in \mathbf{Z}_\alpha \subset \mathbf{T}^{\mathrm{SG}}, \tag{9}$$

i.e. $\mathbf{Z}_\alpha$ is a set of odd-reduced supermatrices parametrized by the even parameter $t \in \Lambda_0$. Then $\mathbf{Z}_\alpha$ is a semigroup under the matrix multiplication ($\alpha \in \Lambda_1$ 'numbers' the semigroups) which is isomorphic to a one-parameter semigroup with the multiplication $\{t_1\} *_\alpha \{t_2\} = \{t_1\}$. This semigroup is called a *right zero semigroup* $\mathcal{Z}_R = \{\bigcup \{t\} ; *_\alpha\}$ and plays an important role (together with the left zero semigroup $\mathcal{Z}_L$ defined in a dual manner) in the general semigroup theory (e.g., see [17], Theorem 1.27, and [18]).

Let

$$B_\alpha (t, u) = \begin{pmatrix} 0 & \alpha t \\ \alpha u & 1 \end{pmatrix} \in \mathbf{B}_\alpha \subset \mathbf{T}^{\mathrm{SG}}, \tag{10}$$

then $\mathbf{B}_\alpha$ is a supermatrix semigroup (numbered by α) which is isomorphic to a two Λ_0-parametric semigroup $\mathcal{B} = \{\bigcup \{t, u\} ; *_\alpha\}$, where the multiplication is $\{t_1, u_1\} *_\alpha \{t_2, u_2\} = \{t_1, u_2\}$. Here every element is an idempotent (as in the previous case too), and so this is a *rectangular band* multiplication [18].

ASSERTION 6. *The one and two parametric subsemigroups of the semigroup of odd-reduced supermatrices* $\mathbf{T}^{\mathrm{SG}}$ *having vanishing Berezinian represent semigroup bands, viz. the left and right zero semigroups and rectangular bands.*

THEOREM 7. *The continuous supermatrix representation of the Rees matrix semigroup over a unit group* $G = e$ *(see* [17,18]) *is given by formulas* (9) *and* (10).

5. Unification of Reduced Supermatrices

Now we try to unify the even- and odd-reduced supermatrices (2) and (3) into a common abstract object. To begin with, consider the multiplication table of all introduced sets

$$\begin{aligned} &\mathbf{D} \cdot \mathbf{D} = \mathbf{D}, && \mathbf{A} \cdot \mathbf{A} = \mathbf{D}, \\ &\mathbf{D} \cdot \mathbf{S} = \mathbf{S}, && \mathbf{T} \cdot \mathbf{A} = \mathbf{S}^{\mathrm{st}}, \\ &\mathbf{S} \cdot \mathbf{D} = \mathbf{S}, && \mathbf{S} \cdot \mathbf{A} = \mathbf{T}^{\Pi}, \\ &\mathbf{A} \cdot \mathbf{T} = \mathbf{S}, && \mathbf{S} \cdot \mathbf{T} = \mathbf{S} \cup \mathbf{T}, \\ &\mathbf{A} \cdot \mathbf{S} = \mathbf{T}, && \mathbf{T} \cdot \mathbf{S} = \mathbf{T}. \end{aligned} \tag{11}$$

Here $\mathrm{st} : \mathrm{Mat}_\Lambda (1|1) \to \mathrm{Mat}_\Lambda (1|1)$ is a supertranspose [12], i.e.

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} a & \beta \\ -\alpha & b \end{pmatrix}.$$

Also, we use the Π-transpose [19] defined by $\Pi\colon \mathrm{Mat}_\Lambda (1|1) \to \mathrm{Mat}_\Lambda (1|1)$ and

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\Pi} = \begin{pmatrix} b & \beta \\ \alpha & a \end{pmatrix}.$$

Note that the sets of supermatrices $\mathbf{S}$ and $\mathbf{T}$ are not closed under st and Π operations, but $\mathbf{S}^{st} \cap \mathbf{S} = \mathbf{D}$ and $\mathbf{T}^{\Pi} \cap \mathbf{T} = \mathbf{A}$.

We observe from the first two relations of (11) that $\mathbf{A}$ plays a role of the left type-changing operator $\mathbf{A} : \mathbf{S} \to \mathbf{T}$ and $\mathbf{A} : \mathbf{T} \to \mathbf{S}$, while $\mathbf{D}$ does not change the type. Next, from the first two relations of (11), it is obviously seen that the sets $\mathbf{S}$ and $\mathbf{D}$ are subsemigroups. Unfortunately, due to the next to last relation of (11), the set $\mathbf{T}$ has no such clear abstract meaning. However, the last relation $\mathbf{T} \cdot \mathbf{S} = \mathbf{T}$ is important from another viewpoint:

THEOREM 8. *Any odd-reduced morphism $\Lambda^{1|1} \to \Lambda^{1|1}$ corresponding to $\mathbf{T}$ can be represented as a product of odd- and even-reduced morphisms, such that*

$$\begin{array}{ccc} & \xrightarrow{\mathbf{S}} & \\ \mathbf{T} & \searrow & \downarrow \mathbf{T} \end{array} \qquad (12)$$

is a commutative diagram.

This decomposition is crucial in applications to the superconformal-like transformation constructions (see [3]).

5.1. REDUCED MATRIX SET SEMIGROUP

To unify the introduced sets (11), we consider the triple products

$$\begin{aligned} \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{T} &= \mathbf{S}, \qquad & \mathbf{T} \cdot \mathbf{A} \cdot \mathbf{T} &= \mathbf{T}, \\ \mathbf{S} \cdot \mathbf{D} \cdot \mathbf{S} &= \mathbf{S}, \qquad & \mathbf{T} \cdot \mathbf{D} \cdot \mathbf{S} &= \mathbf{T}. \end{aligned} \qquad (13)$$

Here we observe that the supermatrices $\mathbf{A}$ and $\mathbf{D}$ play the role of 'sandwich' elements in a special $\mathbf{S}$ and $\mathbf{T}$ multiplication. Moreover, the sandwich elements are in one-to-one correspondence with the right sets on which they act, and so they are 'sensible from the right'. Therefore, it is quite natural to introduce the following

DEFINITION 9. A sandwich product of the reduced supermatrix sets $\mathbf{R} = \mathbf{S}, \mathbf{T}$ is

$$\mathbf{R}_1 \odot \mathbf{R}_2 \stackrel{\text{def}}{=} \begin{cases} \mathbf{R}_1 \cdot \mathbf{D} \cdot \mathbf{R}_2, & \mathbf{R}_2 = \mathbf{S}, \\ \mathbf{R}_1 \cdot \mathbf{A} \cdot \mathbf{R}_2, & \mathbf{R}_2 = \mathbf{T}. \end{cases} \qquad (14)$$

In terms of the sandwich product from (13), we obtain

$$\begin{aligned} \mathbf{S} \odot \mathbf{T} &= \mathbf{S}, \qquad & \mathbf{T} \odot \mathbf{T} &= \mathbf{T}, \\ \mathbf{S} \odot \mathbf{S} &= \mathbf{S},, \qquad & \mathbf{T} \odot \mathbf{S} &= \mathbf{T}. \end{aligned} \qquad (15)$$

PROPOSITION 10. *The $\odot$-multiplication is associative.*

DEFINITION 11. The elements **S** and **T** form a semigroup under $\odot$-multiplication (14), which we call a reduced matrix set semigroup and denote $\mathcal{RMS}_{\text{set}}$.

Comparing (15) with the multiplication of the one-parametric odd-reduced supermatrices (9), we observe

THEOREM 12. *The reduced matrix set semigroup is isomorphic to a special right zero semigroup, i.e.* $\mathcal{RMS}_{\text{set}} \cong \mathcal{Z}_R = \{\mathbf{R} = \mathbf{S}, \mathbf{T}; \odot\}$.

5.2. SCALARS, ANTI-SCALARS, GENERALIZED MODULES AND REDUCED MATRIX SANDWICH SEMIGROUP

Now we introduce an analogue of $\odot$-multiplication for the reduced matrices per se (not for sets). First we define the structure of a generalized Λ-module in $\text{Hom}_0(\Lambda^{1|1}, \Lambda^{1|1})$ in some alternative way, the even part of which is described in [12] (in ordinary matrix theory, it is a trivial fact that a product of a matrix and a number is equal to a product of a matrix and a diagonal matrix having this number on the diagonal).

DEFINITION 13. In $\text{Mat}_\Lambda(1|1)$ a scalar (matrix) $E(x)$ and anti-scalar (matrix) $\mathcal{E}(\chi)$ are defined by

$$
\begin{aligned}
E(x) &\stackrel{\text{def}}{=} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathbf{D} = \text{Mat}_\Lambda^{\text{Diag}}(1|1), \quad x \in \Lambda_0, \\
\mathcal{E}(\chi) &\stackrel{\text{def}}{=} \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix} \in \mathbf{A} = \text{Mat}_\Lambda^{\text{Adiag}}(1|1), \quad \chi \in \Lambda_1.
\end{aligned}
\tag{16}
$$

ASSERTION 14. *The Berezin queer subalgebra*

$$
Q_\Lambda(1) \equiv \begin{pmatrix} x & \chi \\ \chi & x \end{pmatrix} \subset \text{Mat}_\Lambda(1|1)
$$

[11] *is a direct sum of the scalar and anti-scalar*

$$
Q_\Lambda(1) = E(x) \oplus \mathcal{E}(\chi). \tag{17}
$$

ASSERTION 15. *The anti-scalars anticommute* $\mathcal{E}(\chi_1)\mathcal{E}(\chi_2) + \mathcal{E}(\chi_2)\mathcal{E}(\chi_1) = 0$, *and so they are nilpotent.*

PROPOSITION 16. *The structure of the generalized* $\Lambda_0 \oplus \Lambda_1$*-module in* $\text{Hom}_0 \times (\Lambda^{1|1}, \Lambda^{1|1})$ *is defined by the action of the scalars and anti-scalars* (16).

This means that everywhere we exchange the multiplication of supermatrices by even and odd elements from Λ with the multiplication by the scalar supermatrices and anti-scalar ones (16). The relations containing the scalars are well known [12], but we obtain new dual scalars for the anti-scalars. Consider their action on the elements $\mathbf{M} \in \mathrm{Mat}_{\Lambda}(1|1)$ in more detail. First we need the following definition.

DEFINITION 17. Left $\mathcal{P}$ and right $\mathcal{Q}$ anti-transposes are $\mathrm{Hom}_0(\Lambda^{1|1}, \Lambda^{1|1}) \to \mathrm{Hom}_1(\Lambda^{1|1}, \Lambda^{1|1})$ mappings acting on $M \in \mathbf{M}$ as

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\mathcal{P}} = \begin{pmatrix} \beta & b \\ a & \alpha \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{\mathcal{Q}} = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}.$$

COROLLARY 18. *Anti-transposes are square roots of the parity changing operator* Π *in the following sense* $\mathcal{PQ} = \mathcal{QP} = \Pi$.

ASSERTION 19. *Anti-transposes satisfy*

$$\begin{aligned} (\mathcal{E}(\chi) M)^{\mathcal{P}} &= \chi M, & (\mathcal{E}(\chi) M)^{\mathcal{Q}} &= \chi M^{\Pi}, \\ (M \mathcal{E}(\chi))^{\mathcal{P}} &= M^{\Pi} \chi, & (M \mathcal{E}(\chi))^{\mathcal{Q}} &= M \chi. \end{aligned} \tag{18}$$

Thus, the concrete realization of right, left and two-sided generalized $\Lambda_0 \oplus \Lambda_1$-modules in $\mathrm{Hom}_0(\Lambda^{1|1}, \Lambda^{1|1})$ is determined by the actions

$$\begin{aligned} &\mathcal{E}(\chi) M = \chi M^{\mathcal{P}}, \qquad M \mathcal{E}(\chi) = M^{\mathcal{Q}} \chi, \\ &\mathcal{E}(\chi_1) M \mathcal{E}(\chi_2) = \chi_1 M^{\Pi} \chi_2, \end{aligned} \tag{19}$$

together with the standard Λ-module structure [12]

$$\begin{aligned} &E(x) M = x M, \qquad M E(x) = M x, \\ &E(x_1) M E(x_2) = x_1 M x_2. \end{aligned} \tag{20}$$

COROLLARY 20. *The generalized* $\Lambda_0 \oplus \Lambda_1$*-module relations are*

$$\begin{aligned} (E(x) M) N &= E(x)(MN), & (M E(x)) N &= M(E(x) N), \\ M(N E(x)) &= (MN) E(x), & (\mathcal{E}(\chi) M) N &= \mathcal{E}(\chi)(MN), \\ (M \mathcal{E}(\chi)) N &= M(\mathcal{E}(\chi) N), & M(N \mathcal{E}(\chi)) &= (MN) \mathcal{E}(\chi). \end{aligned} \tag{21}$$

where $M, N \in \mathrm{Mat}_{\Lambda}(1|1)$.

DEFINITION 21. The odd scalar and odd anti-scalar are defined by

$$E(\chi) \overset{\mathrm{def}}{=} \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix} \in \mathrm{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right) \tag{22}$$

and

$$\mathcal{E}(x) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \in \text{Hom}_1\left(\Lambda^{1|1}, \Lambda^{1|1}\right), \tag{23}$$

respectively.

PROPOSITION 22. *The structure of the generalized* $\Lambda_0 \oplus \Lambda_1$*-module in* $\text{Hom}_1(\Lambda^{1|1}, \Lambda^{1|1})$ *is determined by the analogous to* (21) *actions of the odd scalar and odd anti-scalar.*

One way to unify the even- (2) and odd-reduced (3) supermatrices into an object analogous to a semigroup, is consideration of a sandwich multiplication similar to (14), but on the level of supermatrices (not sets), by means of the scalars and anti-scalars as sandwich supermatrices. Indeed, the ordinary supermatrix product can be written as $M_1 M_2 = M_1 E(1) M_2$. We cannot find an analogue to this relation for the anti-scalar, because among odd variables $\chi \in \Lambda_1$, there is no unity. Therefore, the only possibility to include $\mathcal{E}(\chi)$ into equal play is consideration of the sandwich elements (16) having arbitrary (or fixed by other special conditions) arguments x and χ. Thus, we naturally come to the following definition.

DEFINITION 23. A sandwich right sensible $\Lambda_0 \oplus \Lambda_1$-product of the reduced supermatrices $R = S,T$ is

$$R_1 \star_X R_2 \stackrel{\text{def}}{=} \begin{cases} R_1 E(x) R_2, & R_2 = S, \\ R_1 \mathcal{E}(\chi) R_2, & R_2 = T, \end{cases} \tag{24}$$

where $X = \{x, \chi\} \in \Lambda_0 \oplus \Lambda_1$.

The $\star_X$-multiplication is associative and its table coincides with (15). Therefore, we have

PROPOSITION 24. *Under the* $\Lambda_0 \oplus \Lambda_1$*-multiplication, the reduced matrices form a semigroup which we call a reduced matrix sandwich semigroup* $\mathcal{RMSS}$.

THEOREM 25. *The reduced matrix sandwich semigroup is isomorphic to a special right zero semigroup, i.e.* $\mathcal{RMSS} \cong \mathcal{Z}_R = \{R = \bigcup S \bigcup T; \star_X\}$.

5.3. DIRECT SUM OF REDUCED SUPERMATRICES

Another way to unify the reduced supermatrices is to consider the connection between them and the generalized $\Lambda_0 \oplus \Lambda_1$-modules.

DEFINITION 26. The reduced supermatrix direct space $\mathcal{RMDS}$ is a direct sum of the even-reduced supermatrix space and the odd-reduced one.

In terms of sets, we have $\mathbf{R}_{\oplus} = \mathbf{S} \oplus \mathbf{T}$. Note that $\mathbf{R}_{\oplus} \neq \mathbf{M}$ because of (5).

ASSERTION 27. *In $\mathcal{RMDS}$, the scalar is Berezin's queer subalgebra $Q_{\Lambda}(1)$ (see (17)).*

THEOREM 28. *In $\mathcal{RMDS}$, the scalar plays the same role for the even-reduced supermatrices as the anti-scalar plays for the odd-reduced ones.*

COROLLARY 29. *The eigenvalues of even- (2) and odd-reduced (3) supermatrices should be found from different equations, viz.*

$$SV = E(x)V, \qquad TV = \mathcal{E}(\chi)V, \tag{25}$$

where V is a column vector, and they are

$$\begin{aligned} x_1 &= a, & x_2 &= b, \\ \chi_1 &= \alpha, & \chi_2 &= \beta. \end{aligned} \tag{26}$$

(see (2) and (3)).

DEFINITION 30. The characteristic functions for the even- and odd-reduced supermatrices are defined in $\mathcal{RMDS}$ by different formulas

$$H_S^{\text{even}}(x) = \text{Ber}\,(E(x) - S), \qquad H_T^{\text{odd}}(\chi) = \text{Ber}\,(\mathcal{E}(\chi) - T). \tag{27}$$

Remark. In the standard Λ-module over $\text{Mat}_{\Lambda}(1|1)$ [11], one derives characteristic functions and eigenvalues for any supermatrix (and for odd-reduced too) from the first equations in (25) and (27) which gives a different result in the odd case (see, e.g., [20]).

Using (2), (3), we easily find

$$H_S^{\text{even}}(x) = \frac{(x-b)(x-a)}{(x-b)^2}, \qquad H_T^{\text{odd}}(\chi) = \frac{(\chi-\beta)(\chi-\alpha)}{b^2}. \tag{28}$$

Here we observe the full symmetry between even- and odd-reduced supermatrices (for this purpose, the cancellation in the first equation was avoided) and consistency with their $\Lambda_0 \oplus \Lambda_1$-eigenvalues (26).

The characteristic polynomial of a supermatrix M is defined by $P_M(M) = 0$ and in complicated cases is constructed from the parts of the characteristic function $H_M(x)$ according to a special algorithm [20] (for a nonsupersymmetric matrix M, it evidently coincides with the characteristic function $P_M(x) = H_M(x) \equiv \det(Ix - M)$, where I is a unity matrix). Due to the existence of zero divisors in Λ, the degree of $P_M(x)$ can be less than $n = p + q$, $M \in \text{Mat}_{\Lambda}(p|q)$. But this algorithm is not directly applicable for the odd-reduced and secondary diagonal supermatrices. As before, we introduce two dual characteristic polynomials and, using (28), obtain the Cayley–Hamilton theorem in $\mathcal{RMDS}$.

THEOREM 31 (The generalized Cayley–Hamilton theorem). *The characteristic polynomials in $\mathcal{RMDS}$ are*

$$
\begin{aligned}
P_S^{\mathrm{even}}(x) &= (x-a)(x-b)\,,\\
P_T^{\mathrm{odd}}(\chi) &= (\chi-\alpha)(\chi-\beta)\,.
\end{aligned}
\tag{29}
$$

and $P_S^{\mathrm{even}}(S)=0$ for any S, but $P_T^{\mathrm{odd}}(T)=0$ for the nilpotent b only.

6. Conclusions

We conclude that almost all the above constructions can be easily extended to arbitrary supermatrices. In the particular case of superconformal-like transformations, it would be interesting to use the alternative reduction introduced here in building the objects analogous to super Riemann or semirigid surfaces, which can also lead to new topological-like models.

References

1. Kobayashi, S.: *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, 1972.
2. Lott, J.: Torsion constraints in supergeometry, *Comm. Math. Phys.* **133** (1990), 563.
3. Duplij, S.: On semigroup nature of superconformal symmetry, *J. Math. Phys.* **32** (1991), 2959.
4. Duplij, S.: On $N = 4$ super Riemann surfaces and superconformal semigroup, *J. Phys. A* **24** (1991), 3167.
5. Duplij, S.: Ideal structure of superconformal semigroups, University of Kaiserslautern, Preprint KL-TH-95/4, CERN-SCAN-9503193, 1995.
6. Crane, L. and Rabin, J. M.: Super Riemann surfaces: uniformization and Teichmüller theory, *Comm. Math. Phys.* **113** (1988), 601.
7. Giddings, S. B. and Nelson, P.: The geometry of super Riemann surfaces, *Comm. Math. Phys.* **116** (1988), 607.
8. Govindarajan, S., Nelson, P., and Wong, E.: Semirigid geometry, *Comm. Math. Phys.* **147** (1992), 253.
9. Alexandrov, M., Kontsevich, M., Schwarz, A., and Zabrodsky, O.: The geometry of the master equation and topological quantum field theory, UC Davis preprint, 1995, hep-th/9502010.
10. Rogers, A.: A global theory of supermanifolds, *J. Math. Phys.* **21** (1980), 1352.
11. Berezin, F. A.: *Introduction to Superanalysis*, D. Reidel, Dordrecht, 1987.
12. Leites, D. A.: Introduction to the theory of supermanifolds, *Russian Math. Surveys* **35** (1980), 1.
13. Ivashchuk, V. D.: Invertibility of elements in infinite-dimensional Grassmann–Banach algebras, *Theoret. Math. Phys.* **84** (1990), 13.
14. Pestov, V.: Soul expansion of G^∞ superfunctions, *J. Math. Phys.* **34** (1993), 3316.
15. Duplij, S.: Some abstract properties of semigroups appearing in superconformal theories, University of Kaiserslautern, Preprint KL-TH-95/11, to appear in *Semigroup Forum*.
16. McAlister, D. B.: Representations of semigroups by linear transformations, 1, 2, *Semigroup Forum* **2** (1971), 189.
17. Clifford, A. H. and Preston, G. B.: *The Algebraic Theory of Semigroups*, Vol. 1, Amer. Math. Soc., Providence, 1961.
18. Howie, J. M.: *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
19. Manin, Y. I.: *Gauge Field Theory and Complex Geometry*, Springer-Verlag, New York, 1988.
20. Urrutia, L. F. and Morales, N.: The Cayley–Hamilton theorem for supermatrices, *J. Phys. A* **27** (1994), 1981.